

An experimental uncertainty implied by failure of the physical Church-Turing thesis

Amir Leshem

*School of Engineering, Bar-Ilan university,
52900, Ramat-Gan, Israel**

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In this paper we prove that given a black box assumed to generate bits of a given non-recursive real Ω there is no computable decision procedure generating sequences of decisions such that if the output is indeed Ω the process eventually accepts the hypothesis while if the output is different than Ω than the procedure will eventually reject the hypothesis from a certain point on. Our decision concept does not require full certainty regarding the correctness of the decision at any point, thus better represents the validation process of physical theories. The theorem has strong implications on the falsifiability of physical theories entailing the failure of the physical Church Turing thesis. Finally we show that our decision process enables to decide whether the mean of an i.i.d. sequence of reals belongs to a specific Δ_2 set of integers. This significantly strengthens the effective version of the Cover-Koplowitz theorem, beyond computable sequences of reals.

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INTRODUCTION

Church-Turing thesis states that every function $f : \mathbb{N} \rightarrow \mathbb{N}$ that is algorithmically computable is recursive or equivalently computable by a universal Turing machine. This is a mathematical statement about algorithms not physical computers. This thesis has been supported by the equivalence of all models of computations to date, e.g., Turing machines, register machines and lambda calculus [3]. A stronger thesis termed the physical Church-Turing thesis is that every finitely realizable physical process can be perfectly simulated by a universal Turing machine operating by finite means (see [2], [9] and the references therein). This thesis is much stronger. It has been argued that there might be finitely realizable physical processes that can compute non-computable functions e.g., by analog computation [15], stronger versions of quantum computing [8],[5] or general relativity [13]. Furthermore [9, 10] claims that some of the open problems in physics (e.g., quantum gravitation) and the failure to generate artificial intelligence might have underlying non-computable physical processes. Similarly the possibility that some physical constants are non-computable has been conjectured. If indeed a physical constant is non-computable then more and more accurate measurements of the constant (or repeated independent observations with i.i.d noise) will provide us more bits of its binary expansion, therefore providing an oracle that breaks the physical Church-Turing thesis.

In this paper we discuss the empirical validation and refutation of the hypothesis that a computing device generates a non-computable real number. Deutsch [2] argues that this is hopeless since any finite experiment generates a finite sequence of measurements with finite precision so the output of any experiment is computable. However, following Cover [1] we might envision a different strategy: Our null hypothesis is that we are given a black box gen-

erating the bits of a specific non-computable real (In order to be able to use the digits provided by the black box we need to know what is the assumed outcome, otherwise a random number generator will provide us almost surely with useless bits of a non-computable real). We now perform an infinite sequence of tests, each sampling more and more bits of the given black box. This measurement procedure serves as an oracle, and our task is to refute the null hypothesis if the oracle is a false oracle. The only requirement we have is that if the oracle is a false oracle (does not provide the bits of Ω) then from a certain point our procedure always rejects the null hypothesis, and if it indeed provides the bits of Ω then from a certain point on we will always accept the hypothesis. This viewpoint is in line with the idea of falsifying a physical theory [12]. Note that while our decisions are asymptotically correct we are never certain about this fact. Since at any finite stage we might change our decision, we never get full certainty regarding the null hypothesis, only growing confidence. This is much like the validation process of physical theories. The theory is accepted only if it is sufficiently simple on one hand and has not yet been refuted by experiment on the other hand.

Cover [1] used this strategy to verify that a mean of a random sequence is rational. His motivation was to provide tests for certain simple representations of well known physical constants [4, 7, 16]. While at every finite stage any confidence interval contains rationals and irrationals he shows that if the number is rational or is irrational outside a certain set of measure 0 then with probability one the sequence of decisions is correct from a certain point on. Koplowitz [6] and later Peres and Dembo [11] proved that if we want to test a hypothesis regarding two sets contained in disjoint F_σ sets then the correct decision is made with probability one for both H_1 and H_0 and the set of measure 0 can be assumed empty. However in all these papers little attention was paid to actual computability of the decision procedures.

The procedures are computable only if the countable sequence of reals is a computable sequence of computable reals.

In this paper we apply the same decision concept to testing the non-computability of physical constants (and actually to any output of a physical black box) providing us with a sequence of approximations to a real number. We show that given the output of a black box which is assumed to provide us bits of a given non-computable real there is NO decision procedure (deterministic or probabilistic) that makes an infinite sequence of decisions such that any false oracle is detected from a certain point on. The proof is general enough to apply to any physical model of computation. It means that if the Church-Turing thesis is true, then there is no way to experimentally refute the hypothesis that a finitely realizable process provides us a non-computable number. This type of uncertainty requires us to choose between falsifiability of such a physical prediction (which would be the most substantial prediction of a theory claiming that a given physical process is non-computable) and our belief in the given theory. Hence accepting the failure of the physical Church-Turing thesis implies that we need to revise the notion of experimental refutation of physical theories as proposed by Popper.

On the positive side we provide a computable decision procedure that almost surely decides asymptotically whether the mean of an i.i.d random process belongs to a given Δ_2 non-computable set of natural numbers using the asymptotic decision concept. Our result significantly strengthens Cover's result since Δ_2 sets cannot be effectively enumerated, and therefore Cover's proof fails to provide a computable decision procedure.

PHYSICAL PROCESSES, DECISION PROCEDURES AND EXPERIMENT DESIGN

In this section we review the concept of an experiment and a decision procedure to falsify a given prediction of a physical theory. We focus on the experimental evidence that the output of a black box is a non-computable real. The formulation is sufficiently general to include statistical decisions as well as deterministic procedures.

First we comment that basically any physical observation is subject to noise and finite accuracy limitations. This implies that the best we can hope from an experimental device is to obtain a sequence of measurements with increasing accuracy rather than a single infinitely accurate measurement. However we do assume that there is no practical limitation on the observation and that by averaging we might reduce effects of observation noise as much as we would like (otherwise it would be trivially impossible to decide anything regarding non-computability, since any finite sequence is by itself computable). Even this assumption ignores some inherent

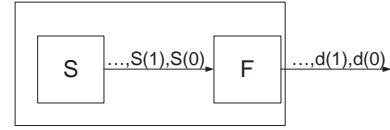


FIG. 1: A general decision process for refuting the null hypothesis $S = \Omega$

limitations on measurement such as the fact that certain parameters cannot be simultaneously measured due to quantum limitations, or fundamental limitations on synchronization, etc.

We now provide some definitions and notations to be used later in the paper.

Definition 1 1. For a finite sequence $s = \langle s(0), s(1), \dots, s(N-1) \rangle$ we denote the length of the sequence by $l(s) = N$.

2. For two sequences s, t where $l(s) \leq l(t)$ we denote $s \leq^* t$ if for all $n < l(s)$ we have $s(n) = t(n)$.

3. Let s be a sequence of integers (finite or infinite). Let $n \leq l(s)$ then $s|n = \langle s(0), s(1), \dots, s(n-1) \rangle$.

4. ω is the set of natural numbers and $2^{<\omega}$ is the binary tree of all binary finite sequences.

Let Ω be a non-computable real. We shall assume without loss of generality that our real Ω is the characteristic function of a non-computable set W . Suppose that as in figure 1 a black box S , provides us at each experiment $n = 0, 1, 2, \dots$ with a bit $S(n)$. We would like at each stage n to decide whether the bits provided by S so far are the initial bits of our real Ω or not. We do not require that all our decisions are correct, however we require that for any $S = \langle S(0), S(1), \dots \rangle$, if $S \neq \Omega$ then from a certain finite stage we decide correctly that $S \neq \Omega$. Similarly if $S = \Omega$ then we do not change our mind regarding this and we accept the null hypothesis from a certain stage onwards. Note that the decision procedure can be viewed as a computable function $F : 2^{<\omega} \rightarrow \{0, 1\}$ where for any $s = \langle s(0), \dots, s(N-1) \rangle$

$$F(s) = 0 \text{ iff we decide that } s \not\leq^* \Omega. \quad (1)$$

Let S be a source of bits. We measure the output of S . A physical theory stating that S generates the bits of a specific non-recursive real is experimentally falsifiable if and only if there is a computable decision procedure $F : 2^{<\omega} \rightarrow \{0, 1\}$ such that for every S , $\lim_{n \rightarrow \infty} F(S|n)$ exists and

$$\lim_{n \rightarrow \infty} F(S|n) = 0 \iff S \neq \Omega. \quad (2)$$

The setup is presented in figure 1. We do not require that we know whether $S \neq \Omega$ or not at any finite stage since

any decision might be reversed after further observation. However we do require that performing more and more measurements of the output of S will eventually accept the null hypothesis

$$H_0 : S = \Omega \quad (3)$$

if it is valid or reject it from a certain point on if $S \neq \Omega$. This notion of verification is consistent with the way we test physical theories in general [12]. We do not know at any finite stage whether the theory is correct but we require that the theory is falsifiable, i.e., if the theory is wrong then there is an experiment falsifying the theory. If the theory is false then from a certain point on our experiments will refute the theory.

MAIN THEOREM

The asymptotic verification described above can be utilized in all existing physical theories e.g., verifying that the absolute zero is not achievable. This is not the case with non-recursiveness. Our main theorem proves that the non-recursiveness of the outcome of a finitely realizable physical system implies a significant measurement uncertainty in the sense that no experiment can be designed to eventually refute or accept the prediction of the theory. Interestingly theorem 4 provides an example of asymptotic decision process regarding membership in Δ_2 sets. Hence not all is lost regarding testing hypotheses about non-recursive sets.

Theorem 2 *Let W be a non-recursive set (e.g., the set of Gödel numbers of all halting Turing machines). Let $\Omega = 1_W$ be the characteristic function of W . Then there is no computable decision procedure $F : 2^{<\omega} \rightarrow \{0, 1\}$ such that $\lim_{n \rightarrow \infty} F(S|n)$ exists and*

$$\lim_{n \rightarrow \infty} F(S|n) = 0 \iff S \neq \Omega \quad (4)$$

Proof: Assume towards contradiction that we are given a recursive $F : 2^{<\omega} \rightarrow \{0, 1\}$ such that (4) holds.

Claim 3 *For every t define*

$$R_t = \{r | t \leq^* r \wedge F(r) = 1\}.$$

If $t \not\leq^ \Omega$ then R_t is finite.*

Proof of the claim: Assume that $t \not\leq^* \Omega$ and R_t is infinite. Define a subtree T_t of $2^{<\omega}$ by

$$T_t = \{s | t \leq^* s \wedge \exists r \in R_t (s \leq^* r)\}.$$

This is the subtree spanned by all extensions of t that are decided the wrong way (since $t \not\leq^* \Omega$). Since R_t is infinite T_t is an infinite tree with finite branching. Therefore by König's lemma it has an infinite branch b . However for

every node $s \leq^* b$ there is an extension $r \in b$ satisfying $F(r) = 1$. Therefore

$$\lim_{n \rightarrow \infty} F(b|n) \neq 0$$

contradicting the definition of F , since if S generates b either we asymptotically accept H_0 , i.e., we decide $b = \Omega$, which is false, or we change our decision infinitely many times. This contradicts the definition of F and the claim is proved.

We are now in position to finish the proof of theorem 2. Let m_0 be sufficiently large so that $F(\Omega|m) = 1$ for all $m_0 \leq m$. Such an m_0 exists since by (4). By our claim we have

$$t \leq^* \Omega \iff \forall m (m_0 \leq m \rightarrow \exists s \in 2^m (t \leq^* s \wedge F(s) = 1)).$$

Let $m_1 = \max\{n, m_0\}$. Then for all n

$$n \in W \iff \forall m (m_1 \leq m \rightarrow \exists s \in 2^m (F(s) = 1) \wedge s(n) = 1).$$

Similarly for all n

$$n \notin W \iff \forall m (m_1 \leq m \rightarrow \exists s \in 2^m (F(s) = 1) \wedge s(n) = 0).$$

Since all internal quantifiers are bounded W is both a Π_1 and Σ_1 set so it is recursive contradicting our assumption.

DECIDING THAT A MEAN BELONGS TO A NON-COMPUTABLE SET

While in the previous section we proved that even given a deterministic output of a black box we cannot define a computable sequence of decisions asymptotically deciding whether the black box provides us the digit of a specific non-computable real, we show that certain decisions regarding non-computable sets can be made. The general properties of Δ_2 sets can be found in [14].

Theorem 4 *Let $A \subseteq \mathbb{N}$ be a Δ_2 set of natural numbers. Assume x_1, x_2, \dots is a sequence of i.i.d real random variables with mean μ and variance $\sigma^2 < \infty$. Then there exist a recursive decision procedure $F : R^{<\omega} \rightarrow \{0, 1\}$ such that almost surely (with respect to realizations of the sequence)*

$$\lim_{n \rightarrow \infty} F(\langle x_1, \dots, x_n \rangle) = 1 \iff \mu \in A \quad (5)$$

Note that unlike the Cover Koplowitz theorem we do not require that the set A admits a recursive enumeration. This strengthens the effective version of the Cover-Koplowitz theorem to Δ_2 sets of integers.

Proof: Our decision procedure consists of two steps. We First decide whether μ is an integer based on an approach similar to the Cover-Koplowitz theorem using that the set \mathbb{N} admits a recursive numeration. Then we

rely on the law of iterated logarithms to obtain an estimate of μ and exploit the definition of A as a Δ_2 set. To test whether $\mu \in \mathbb{N}$ let $\bar{x}_N = \frac{1}{N} \sum_{n=1}^N x_n$ be the sample mean and let $\bar{\sigma}_N^2 = \frac{1}{N} \sum_{n=1}^N (x_n - \bar{x}_N)^2$ be the sample variance. Almost surely $\bar{x}_N \rightarrow \mu$ and $\bar{\sigma}_N^2 \rightarrow \sigma^2$. Furthermore by the law of iterated logarithms almost surely only finitely many times $|\mu - \bar{x}_N| > \sqrt{\frac{\sigma^2 \log \log N}{N}}$ and for every $\varepsilon > 0$ almost surely only finitely many times

$$\sigma^2 > (1 + \varepsilon) \bar{\sigma}_N^2 \quad (6)$$

Fix $\varepsilon > 0$. Let $\delta_N = \sqrt{\frac{(1+\varepsilon)\bar{\sigma}_N^2 \log \log N}{N}}$ and $\hat{\mu}_N = \text{round}(\bar{x}_N)$. If $\mu \in \mathbb{N}$ then with probability one only finitely many times $\hat{\mu}_N \neq \mu$, and except finitely many times this holds whenever $|\hat{\mu}_N - \bar{x}_N| < \delta_N$. If on the other hand $\mu \notin \mathbb{N}$ then almost surely except finitely many N 's $|\mu - \hat{\mu}_N| > \delta_N$ and therefore almost surely except finitely many times $|\hat{\mu}_N - \bar{x}_N| > \delta_N$. Let $\langle d_n : n = 0, 1, \dots \rangle$ be a sequence of decisions for the hypothesis $H_0^{\mathbb{N}} : \mu \in \mathbb{N}$ given by

$$d_n = \begin{cases} 1 & \text{if } |\hat{\mu}_n - \bar{x}_n| \leq \delta_n \\ 0 & \text{if } |\hat{\mu}_n - \bar{x}_n| > \delta_n \end{cases} \quad (7)$$

By the discussion above almost surely d_n is correct except finitely many times. If $d_n = 0$ decide $\mu \notin A$. Otherwise we assume that $\mu \in \mathbb{N}$ and almost surely $\hat{\mu}_n = \mu$ for all but finitely many n 's. Hence if $d_n = 1$ assume that $\mu_n \in \mathbb{N}$. We now define a procedure for testing whether $\hat{\mu}_n \in A$. To that end recall that since A is Δ_2 there are recursive relations $\phi(m, k, n), \psi(m, k, n)$ such that

$$\begin{aligned} n \in A &\iff \exists m \forall k [\phi(m, k, n)] \\ n \notin A &\iff \exists m \forall k [\psi(m, k, n)] \end{aligned} \quad (8)$$

Definition 5 An m_0 such that $\forall k \phi(m_0, k, n)$ is called a witness for $n \in A$. An m_0 such that $\forall k \psi(m_0, k, n)$ is called a witness for $n \notin A$. m_0 decides $n \in A$ if m_0 is a witness for $n \in A$ or for $n \notin A$.

Obviously by (8) for each n either there is a witness that $n \in A$ or there is a witness for $n \notin A$. Therefore given n and m if we compute $\langle (\phi(m, k, n), \psi(m, k, n)), k = 1, 2, \dots \rangle$ we must obtain some k where either $\phi(m, k, n)$ fails or $\psi(m, k, n)$ fails. Moreover by definition there is a least m where one of the formulas holds for all k .

Let $\hat{\mu}_N$ be the estimate of μ given that $d_N = 1$. Let m_0 be the minimal m such that only one of the formulas fails for some k (assume without loss of generality this is ϕ). Then with probability one except finitely many N 's

$$\forall k < N [\phi(m_0, k, \hat{\mu}_N)] \quad (9)$$

and for sufficiently large N m_0 is the minimal one. This suggests the following decision procedure: Assume that

we have decided that at stage N that $\mu \in \mathbb{N}$ ($d_N = 1$). Decide $e_N = 1$ if the minimal m such that

$$\forall k < N [\phi(m, k, \hat{\mu}_N)] \quad (10)$$

is smaller than the minimal m such that

$$\forall k < N [\psi(m, k, \hat{\mu}_N)]. \quad (11)$$

By the discussion above, if $\mu \in A$ then $e_N = 1$ except finitely many N 's and if $\mu \notin A$, $e_N = 0$ except finitely many N 's. This ends the proof.

Finally we comment that if we replace \mathbb{N} by any countable recursive sequence of recursive reals $S = \langle s_n : n \in \mathbb{N} \rangle$ and $A \subseteq \mathbb{N}$ is Δ_2 then we can extend the above result to test whether $\mu \in \langle s_n : n \in A \rangle$. However similarly to Cover's result if the closure of S is uncountable then there is a measure 0 subset of $\mathbb{R} \setminus S$ on which the test might fail.

* URL: <http://www.eng.biu.ac.il/~leshema>

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